

# Explicit description of generic representations for quivers of type $\mathbb{A}_n$ or $\mathbb{D}_n$

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## Abstract

We describe explicitly a generic representation for Dynkin quivers of type  $\mathbb{A}_n$  or  $\mathbb{D}_n$  for any dimension vector.

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## 1. Introduction

Let  $Q$  be a Dynkin quiver of type  $\mathbb{A}_n$  or  $\mathbb{D}_n$ , i.e. the underlying graph of  $Q$  is either

$$1 \xrightarrow{\alpha_1} 2 \xrightarrow{\alpha_2} 3 \quad \dots \quad n-1 \xrightarrow{\alpha_{n-1}} n$$

or

$$\begin{array}{c} 1 \\ \searrow \alpha_1 \\ 3 \xrightarrow{\alpha_3} 4 \\ \nearrow \alpha_2 \\ 2 \end{array} \quad \dots \quad n-1 \xrightarrow{\alpha_{n-1}} n$$

We denote the category of finite dimensional representations of  $Q$  by  $\text{rep } Q$ , and for a dimension vector  $\mathbf{d} = (d_1, \dots, d_n) \in \mathbb{N}^n$  we consider the space  $\text{rep}(Q, \mathbf{d})$  of representations  $X$  of  $Q$  over an algebraically closed field  $k$  with  $X(i) = k^{d_i}$  for any vertex  $i$  of  $Q$ . The affine algebraic group  $\text{Gl}(\mathbf{d}) = \prod_{i=1}^n \text{Gl}(d_i)$  acts on the affine space

$$\text{rep}(Q, \mathbf{d}) = \prod_{i=1}^{n-1} \text{Mat}(d_{t\alpha_i} \times d_{s\alpha_i}, k)$$

by  $(g \star X)(\alpha_i) = g(t\alpha_i)X(\alpha_i)g(s\alpha_i)^{-1}, i = 1, \dots, n-1$ , for  $g = (g_1, \dots, g_n) \in \text{Gl}(\mathbf{d})$  and  $X \in \text{rep}(Q, \mathbf{d})$ , where  $s\alpha$  and  $t\alpha$  denote the starting and terminating vertex of the arrow  $\alpha$ . Note that orbits in  $\text{rep}(Q, \mathbf{d})$  are just isomorphism classes of representations.

As  $Q$  is of finite representation type [? ],  $\text{rep}(Q, \mathbf{d})$  contains finitely many orbits and thus a unique dense orbit  $\text{Gl}(\mathbf{d})T$  with respect to the Zariski topology. A representation in the dense orbit is called generic; it is unique up to isomorphism. By [? ],  $T$  is generic if and only if  $\text{Ext}^1(T, T) = 0$ . Although "most" representations in  $\text{rep}(Q, \mathbf{d})$  are generic, it is not obvious to exhibit a generic representation explicitly. Our goal in this paper is to do this for Dynkin quivers of type  $\mathbb{A}_n$  and  $\mathbb{D}_n$ . In fact, the case  $\mathbb{A}_n$  has already been treated in Abeasis [? ]. A different description from the one presented here in case  $\mathbb{D}_n$  as well as several examples (but no proof) can be found in [? ].

As a byproduct of our description, we obtain a decomposition of  $T$  as a direct sum of indecomposable representations  $T_j$  and thus the canonical decomposition of  $\mathbf{d}$  [? ]. There are algorithms to determine the canonical decomposition of  $\mathbf{d} = \sum \mathbf{d}_j$  for any quiver [? ], [? ], and it is easy to write down matrices  $T_j(\alpha_i)$ , for  $i = 1, \dots, n-1$ , that define an indecomposable representation  $T_j$  in  $\text{rep}(Q, \mathbf{d}_j)$  if  $Q$  is of type  $\mathbb{A}_n$  or  $\mathbb{D}_n$ . So this gives another way of exhibiting explicit matrices for a representation  $T$  in the open orbit of  $\text{rep}(Q, \mathbf{d})$ . Our approach has the merit of producing a matrix  $T(\alpha_i)$  that only depends on  $d_i$  and  $d_{i+1}$ , the dimensions in the starting and terminating vertex of  $\alpha_i$ , for all arrows if  $Q$  is of type  $\mathbb{A}_n$ , and for all but  $\alpha_1$  and  $\alpha_2$  in case  $Q$  is of type  $\mathbb{D}_n$ . In the latter case, the matrices of  $T$  on the remaining two arrows depend on the dimensions of the vector spaces in the three vertices in which one of the two arrows starts or ends.

It is possible but combinatorially more involved to apply the method presented here for Dynkin quivers of type  $\mathbb{E}_n, n = 6, 7, 8$ . In fact, any quiver  $Q$  of type  $\mathbb{E}_n$  can be obtained from a quiver  $Q'$  of type either  $\mathbb{A}_{n-1}$  or  $\mathbb{D}_{n-1}$  by adding one vertex and one arrow. The problem is that the partially ordered set of isomorphism classes of indecomposable representations  $U$  of  $Q'$ , ordered by

the existence of a non-zero homomorphism, is more complicated.

The article is accessible for mathematicians with little knowledge in representation theory. In fact, we only use that representations of a quiver without oriented cycles form an abelian category of global dimension at most one and some homological algebra, but no Auslander-Reiten theory. Even Gabriel's theorem [?] mentioned earlier is not necessary.

## 2. Results

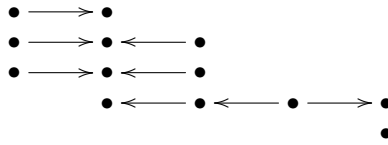
**Proposition 2.1.** *For a Dynkin quiver  $Q$  of type  $\mathbb{A}_n$  and a dimension vector  $\mathbf{d} = (d_1, \dots, d_n)$  the following representation  $T = T_{Q, \mathbf{d}} \in \text{rep}(Q, \mathbf{d})$  is generic: Choose for  $T(\alpha_i)$  the matrix consisting of the identity matrix  $\mathbf{1}_{\min\{d_i, d_{i+1}\}}$  of largest possible size in its upper left or lower right corner if  $\alpha_i$  points to the right or left, respectively, and zeros otherwise.*

Explicitly, we choose

$$T(\alpha_1) = \begin{pmatrix} \mathbf{1}_{d_1} \\ 0 \end{pmatrix} \text{ if } d_1 \leq d_2 \text{ and } T(\alpha_1) = \begin{pmatrix} \mathbf{1}_{d_2} & 0 \end{pmatrix} \text{ if } d_1 \geq d_2$$

for  $\alpha_1 : 1 \rightarrow 2$ . As an example, we draw  $T_{Q, \mathbf{d}}$  for

$$Q = 1 \longrightarrow 2 \longleftarrow 3 \longleftarrow 4 \longrightarrow 5, \mathbf{d} = (3, 4, 3, 1, 2) :$$



Each bullet represents a basis vector. An arrow starting from a bullet means that the basis vector at the start of the arrow is mapped to the one at the terminus. If no arrow starts at a bullet, the corresponding basis vector is mapped to zero.

**Proposition 2.2.** *Let  $Q$  be a Dynkin quiver of type  $\mathbb{D}_n, n \geq 4$ , with the arrows  $\alpha_1$  and  $\alpha_2$  both starting at the vertex 3, and let  $\mathbf{d} = (d_1, \dots, d_n)$  be a dimension*

vector with  $d_1 \leq d_2 \leq d_3$ . The following representation  $T = T_{Q, \mathbf{d}} \in \text{rep}(Q, \mathbf{d})$  is generic:

1. Choose for  $T(\alpha_i)$  the matrix consisting of the identity matrix  $\mathbf{1}_{\min\{d_i, d_{i+1}\}}$  of largest possible size in its upper left or lower right corner if  $\alpha_i$  points to the right or left, respectively, and zeros otherwise,  $i = 3, \dots, n-1$ .
2. If  $d_1 + d_2 \leq d_3$ , set  $d'_3 = d_3 - d_1 - d_2$  and

$$T(\alpha_1) = \begin{pmatrix} 0 & \mathbf{1} & 0 & S \end{pmatrix} : k^{d'_3} \oplus k^{d_1} \oplus k^{d_2-d_1} \oplus k^{d_1} \rightarrow k^{d_1},$$

$$T(\alpha_2) = \begin{pmatrix} 0 & 0 & \mathbf{1} & 0 \\ 0 & 0 & 0 & \mathbf{1} \end{pmatrix} : k^{d'_3} \oplus k^{d_1} \oplus k^{d_2-d_1} \oplus k^{d_1} \rightarrow k^{d_2-d_1} \oplus k^{d_1},$$

3. If  $d_1 + d_2 \geq d_3$ , set  $d'_2 = d_3 - d_2$ ,  $d'_3 = d_1 + d_2 - d_3$  and

$$T(\alpha_1) = \begin{pmatrix} \mathbf{1} & 0 & S & 0 \\ 0 & 0 & 0 & \mathbf{1} \end{pmatrix} : k^{d'_2} \oplus k^{d_2-d_1} \oplus k^{d'_2} \oplus k^{d'_3} \rightarrow k^{d'_2} \oplus k^{d'_3},$$

$$T(\alpha_2) = \begin{pmatrix} 0 & \mathbf{1} & 0 & 0 \\ 0 & 0 & \mathbf{1} & 0 \\ 0 & 0 & 0 & \mathbf{1} \end{pmatrix} : k^{d'_2} \oplus k^{d_2-d_1} \oplus k^{d'_2} \oplus k^{d'_3} \rightarrow k^{d_2-d_1} \oplus k^{d'_2} \oplus k^{d'_3},$$

where  $S = (\sigma_{ij}) \in \text{Mat}(m \times m, k)$  is the matrix with  $\sigma_{ij} = \delta_{i, m+1-j}$  and  $\mathbf{1}$  is the identity matrix of suitable size.

Note that in case  $d_1 > d_3$ , a generic representation is the direct sum of a generic representation with dimension vector  $\mathbf{d} - (d_1 - d_3)\mathbf{dim}S_1$  with  $d_1 - d_3$  copies of  $S_1$ , where  $S_1$  denotes the one-dimensional representation supported at the vertex 1.

Generic representations for other orientations of  $Q$  or other dimension vectors are obtained from the given ones by duality, using the symmetry exchanging the vertices 1 and 2 if possible, or by applying the "reflection functor" [?] corresponding to the vertices 1 or 2. Indeed, suppose  $\alpha_1$  points to

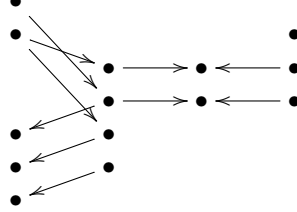
the right in  $Q$ , let  $Q'$  be obtained from  $Q$  by reversing  $\alpha_1$ , and fix a dimension vector  $\mathbf{d}'$  for  $Q'$  with  $d'_1 \leq d'_3$ . Then it is easy to check that a representation  $X' \in \text{rep}(Q', \mathbf{d}')$  is generic if and only if the representation  $X$  of  $Q$  obtained from  $X'$  by replacing  $X'(\alpha'_1) : X'(1) \rightarrow X'(3)$  by the inclusion  $X(\alpha_1) : \ker X'(\alpha'_1) \rightarrow X'(3)$  (without changing any other vector spaces or arrows) and by choosing  $\mathbf{d} = (d'_3 - d'_1, d'_2, \dots, d'_n)$  is generic in  $\text{rep}(Q, \mathbf{d})$ .

Note that we could fix the direction of  $\alpha_{n-1}$  as well by a reflection, but this would limit us to  $d_{n-1} \geq d_n$ , an inconvenience for our inductive proofs.

Considering the example

$$Q = \begin{array}{ccccc} & 1 & & & \\ & \searrow & & & \\ & & 3 & \longrightarrow & 4 \longleftarrow 5, \\ & \swarrow & & & \\ 2 & & & & \end{array}, \quad \mathbf{d} = (2, 3, 4, 2, 3),$$

we find



for the generic representation.

### 3. Proofs

We will prove both results by induction on  $n$ . We denote the one-dimensional representation supported at  $n$  by  $S$ , and we let  $Q'$  and  $\mathbf{d}'$  be the full subquiver containing all vertices of  $Q$  except  $n$  and the restriction of  $\mathbf{d}$  to  $Q'$ , respectively. For a representation  $X$  of  $Q'$ , we still denote by  $X$  the representation of  $Q$  obtained by "extension by zero", i.e. by setting  $X(n) = 0, X(\alpha_{n-1}) = 0$ .

Without loss of generality we may suppose that the arrow  $\alpha_{n-1}$  starts at  $n-1$ : Consider the opposite quiver if it does not.

The following lemma and its dual are the key to the proofs. For two representations  $X$  and  $Y$  of a quiver  $Q$ , we set

$$[X, Y] = \dim_k \operatorname{Hom}_Q(X, Y), \quad {}^1[X, Y] = \dim_k \operatorname{Ext}_Q^1(X, Y).$$

Let  $Q$  be a quiver with a sink  $s$ , and denote by  $Q'$  the full subquiver containing all vertices except  $s$ . Let  $S$  be the 1-dimensional representation of  $Q$  supported at  $s$ , and note that  $S$  is projective. For a representation  $X$  of  $Q$  with  ${}^1[X, S] = 1$ , denote by  $\tilde{X}$  the middle term of a non-split short exact sequence

$$\sigma_X : 0 \rightarrow S \rightarrow \tilde{X} \rightarrow X \rightarrow 0$$

and note that  $\tilde{X}$  is well-defined up to isomorphism. We set  $\tilde{X} = 0$  for  $X = 0$ .

**Lemma 3.1.** *Keeping the notations just introduced, let  $U, V, W$  be representations of  $Q'$  for which  $U \oplus V \oplus W$  is generic. Suppose moreover that  ${}^1[W, S] = 0$  and  $[U, U] = {}^1[U, S] = 1$  and  $[V, V] = {}^1[V, S] = 1$ , respectively, in case  $U$  and  $V$  are non-zero. Note that  $V$  may be equal to  $U$ .*

*Then we have*

1.  $[U \oplus V, \tilde{U} \oplus \tilde{V}] = 0$ ,
2.  $\tilde{U}$  and  $\tilde{V}$  are indecomposable,  ${}^1[\tilde{U}, V \oplus W \oplus \tilde{V} \oplus S] = 0$ ,  ${}^1[U \oplus V \oplus W, \tilde{U}] = 1 - [V, U]$ ,  $[\tilde{U}, \tilde{V}] = [\tilde{U}, V] = [U, V]$ , and  $[S, \tilde{U}] = 1$ ,
3. In case  $[V, U] = 1$ , the representations  $\tilde{U} \oplus U \oplus V \oplus W, \tilde{U} \oplus \tilde{V} \oplus V \oplus W$  and  $\tilde{U} \oplus \tilde{V} \oplus S \oplus W$  are generic.

*Proof.* Note that  $U$  and  $V$  are indecomposable as their endomorphism algebras are one-dimensional. In order to prove our first claim, we show that, for an indecomposable representation  $X$  of  $Q'$ , the conditions  ${}^1[X \oplus U, X \oplus U] = 0$ ,  ${}^1[X, S] = 1$  imply  $[X, \tilde{U}] = 0$ .

Mapping  $X$  to  $\sigma_U$ , we obtain the exact sequence

$$\begin{aligned} 0 \rightarrow \operatorname{Hom}(X, S) = 0 \rightarrow \operatorname{Hom}(X, \tilde{U}) \rightarrow \operatorname{Hom}(X, U) \rightarrow \\ \rightarrow \operatorname{Ext}^1(X, S) = k \rightarrow \operatorname{Ext}^1(X, \tilde{U}) \rightarrow \operatorname{Ext}^1(X, U) = 0 \rightarrow 0. \end{aligned}$$

In case  $[X, U] = 0$ , we obviously have  $[X, \tilde{U}] = 0$ . Suppose  $[X, U] \neq 0$  and fix a non-zero homomorphism  $f : X \rightarrow U$ . It is enough to prove that the connecting homomorphism  $\text{Hom}(X, U) \rightarrow \text{Ext}^1(X, S)$  is injective, or equivalently that the pull-back of  $\sigma_U$  by  $f$  does not split. From  ${}^1[X, S] = \sum \dim X(s\alpha)$ , we conclude that

$$\sum \dim X(s\alpha) = \sum \dim U(s\alpha) = 1,$$

where all sums range over all arrows  $\alpha$  ending at the sink  $s$ . As  ${}^1[U, X] = 0$ , we know from [?] that  $f$  is either a monomorphism or an epimorphism, and thus there is an arrow  $\alpha : t \rightarrow s$  for which  $X(t) = U(t) = k, f(t) \neq 0$ , and therefore  $f$  cannot factor through the projection  $\tilde{U} \rightarrow U$ . This implies that indeed the pull-back of  $\sigma_U$  by  $f$  does not split.

All other claims now follow from considering the long exact sequences obtained by mapping either  $\sigma_U$  or  $\sigma_V$  to suitable representations or by mapping suitable representations to either  $\sigma_U$  or  $\sigma_V$ .  $\square$

We are ready to prove proposition ?? : We have defined  $T$  in such a way that there is a decomposition  $T = (\oplus_{j=1}^{d_n} T_j) \oplus R$  with  $T_j$  indecomposable for all  $j$ ,  $R(n) = 0$  and such that  $T_j(n) = ke_j$ , where  $e_1, \dots, e_{d_n}$  is the standard basis of  $T(n) = k^{d_n}$ . We will show by induction on  $n$  that  ${}^1[T, T] = 0$ , that in addition  $[T_j, T_l] \leq 1$  for all  $j, l$  and that the isomorphism classes in  $\{T_j : j = 1, \dots, d_n\}$  are linearly ordered by the existence of non-zero homomorphisms, i.e. that for  $T_j$  and  $T_l$  non-isomorphic we have  $[T_j, T_l] = 1$  if and only if  $j > l$ . Note that these conditions are trivially satisfied for  $n = 1$  as in this case all indecomposable direct summands of  $T$  are isomorphic and there are no non-trivial extensions.

Remember that  $Q'$  is the full subquiver of  $Q$  containing all vertices except  $n$ , and consider the representation  $T' = T_{Q', \mathbf{d}'}$  of  $Q'$ , which is just the restriction of  $T = T_{Q, \mathbf{d}}$  to  $Q'$ . By induction we know that  ${}^1[T', T'] = 0$ , that  $T'$  has a decomposition  $T' = (\oplus_{j=1}^{d_{n-1}} T'_j) \oplus R'$  with  $T'_j$  indecomposable,  $R'(n-1) = 0$ , that  $[T'_j, T'_l] \leq 1$ , and that for  $T'_j$  and  $T'_l$  non-isomorphic we have  $[T'_j, T'_l] = 1$  if and only if  $j > l$  for  $1 \leq j, l \leq d_{n-1}$ .

In case  $d_n \geq d_{n-1}$ , we set  $R = R', T_j = \tilde{T}'_j$  for  $j = 1, \dots, d_{n-1}$  and  $T_j = S$

for  $j = d_{n-1} + 1, \dots, d_n$ . All conditions needed for  $T$  then follow from Lemma ?? . If  $d_n \leq d_{n-1}$ , we set  $R = \oplus_{j=d_n+1}^{d_{n-1}} T'_j \oplus R'$  and  $T_j = \widetilde{T}'_j$  for  $j \leq d_n$  . As  ${}^1[R', S] = 0, {}^1[T'_j, S] = 1$  for  $j \leq d_{n-1}$  and  $[T'_j, T'_l] = 1$  for  $l \leq d_n < j \leq d_{n-1}$  , Lemma ?? yields that  $T$  satisfies all conditions stated.

By our inductive procedure we obtain in addition a decomposition of  $T$  as a direct sum of indecomposables. Indeed, the summands  $T'_j$  of  $T'$  with  $d_n < j \leq d_{n-1}$  are indecomposable direct summands of  $T$  that will not be modified in subsequent steps.

Let us turn to the proof of proposition ?? . We use the same idea as for  $\mathbb{A}_n$ , but if  $Q$  is of type  $\mathbb{D}_n$ , the pairwise non-isomorphic indecomposables whose support contains the vertex  $n$  are not linearly ordered by the existence of a non-zero homomorphism. Indeed, there is no non-zero homomorphism between the indecomposables with dimension vectors  $(1, 0, 1, \dots, 1)$  and  $(0, 1, 1, \dots, 1)$ . It will turn out that these two are the only "incomparable" ones, though. We first prove a supplement to lemma ??, treating this case.

**Lemma 3.2.** *Keeping the notations and hypotheses from lemma ?? we assume in addition that  $[U, V] = [V, U] = 0$ .*

1. *We have  ${}^1[V, \tilde{U}] = {}^1[U, \tilde{V}] = 1$ .*
2. *The middle term  $X$  of any non-split short exact sequence*

$$\tau_{U,V} : 0 \rightarrow \tilde{U} \rightarrow X \rightarrow V \rightarrow 0$$

*satisfies  $[X, X] = 1, {}^1[X, X] = 0$  and is thus indecomposable and generic.*

3. *The dimensions of the only non-zero homomorphism spaces between any two distinct representations among  $U, \tilde{U}, V, \tilde{V}, X$  are  $[\tilde{U}, U] = [\tilde{V}, V] = [X, U] = [X, V] = [\tilde{U}, X] = [\tilde{V}, X] = 1$ .*
4. *The dimensions of the only non-zero extension spaces between any two representations among  $U, \tilde{U}, V, \tilde{V}, X$  are  ${}^1[V, \tilde{U}] = {}^1[U, \tilde{V}] = 1$ . The middle term of a non-split exact sequence in either one of the spaces  $\text{Ext}^1(V, \tilde{U})$  or  $\text{Ext}^1(U, \tilde{V})$  is isomorphic to  $X$ .*



*Proof.* Only the very last statement requires a new idea. Let  $Y$  be the middle term of a non-split exact sequence in  $\text{Ext}^1(U, \tilde{V})$ . Note that

$$\mathbf{dim} Y = \mathbf{dim}(U \oplus \tilde{V}) = \mathbf{dim}(U \oplus V \oplus S) = \mathbf{dim}(V \oplus \tilde{U}) = \mathbf{dim} X.$$

We know that  $X$  is generic, and, exchanging  $U$  and  $V$ , we see that  $Y$  is generic as well. But the generic representation is unique up to isomorphism.  $\square$

From now on we assume  $Q$  to be of type  $\mathbb{D}_n, n \geq 4$ , with both arrows  $\alpha_1$  and  $\alpha_2$  starting at the vertex 3 and  $\alpha_{n-1}$  starting at the vertex  $n-1$ . We fix  $\mathbf{d} \in \mathbb{N}^n$  with  $d_1 \leq d_2 \leq d_3$ . As a first step, we reformulate Lemma ?? for a quiver of type  $\mathbb{D}_n$ . For  $n \geq 3$ , we set

$$\tilde{Y}_+ = T_{Q, (1, 0, 1, \dots, 1, 1)}, \tilde{Y}_- = T_{Q, (0, 1, 1, \dots, 1, 1)},$$

and for  $n \geq 4$  moreover

$$Y_+ = T_{Q, (1, 0, 1, \dots, 1, 0)}, Y_- = T_{Q, (0, 1, 1, \dots, 1, 0)}.$$

Note that, for  $n \geq 4$ ,  $\tilde{Y}_\pm$  is indeed the middle term of a non-split short exact sequence in  $\text{Ext}^1(Y_\pm, S)$ . By  $X$  we denote the middle term of  $\tau_{Y_+, Y_-}$  or  $\tau_{Y_-, Y_+}$ .

**Corollary 3.3.** 1. *The dimensions of the only non-zero homomorphism spaces between any two distinct representations among  $Y_+, \tilde{Y}_+, Y_-, \tilde{Y}_-, X$  are*

$$[\tilde{Y}_+, Y_+] = [\tilde{Y}_-, Y_-] = [X, Y_+] = [X, Y_-] = [\tilde{Y}_+, X] = [\tilde{Y}_-, X] = 1.$$

2. *The dimensions of the only non-zero extension spaces between any two distinct representations among  $Y_+, \tilde{Y}_+, Y_-, \tilde{Y}_-, X$  are*

$${}^1[Y_-, \tilde{Y}_+] = {}^1[Y_+, \tilde{Y}_-] = 1.$$

*The middle term of a non-split exact sequence in either one of the spaces  $\text{Ext}^1(Y_-, \tilde{Y}_+)$  or  $\text{Ext}^1(Y_+, \tilde{Y}_-)$  is isomorphic to  $X$ .*

Corollary ?? allows us to prove that  $T_{Q, \mathbf{d}}$  is generic for special  $\mathbf{d}$ :

**Lemma 3.4.** 1. For  $n \geq 3$ ,  $r, s \geq 0$  and  $\mathbf{e}(r, s) = (r, r+s, 2r+s, \dots, 2r+s)$ , the representation  $T_{Q, \mathbf{e}(r, s)}$  has a decomposition  $T_{Q, \mathbf{e}(r, s)} = \bigoplus_{j=1}^{2r+s} T_j$  with  $T_j$  indecomposable and

$$T_j(n) = \begin{cases} ke_j & \text{for } j \leq r+s, \\ k(e_j - e_{j'}) & \text{for } r+s < j \leq 2r+s, \end{cases}$$

where  $j' = 2r+s+1-j$ . Moreover,  $T_j$  equals  $\tilde{Y}_+$  for  $j \leq r$  and  $\tilde{Y}_-$  for  $j > r$ . In particular,  $T_{Q, \mathbf{e}(r, s)}$  is generic.

2. Suppose  $n \geq 4$ ,  $r, s \geq 0$ ,  $0 \leq u \leq 2r+s$ . For  $j > r+s$ , we set  $j' = 2r+s+1-j$ . For  $\mathbf{e}(r, s, u) = (r, r+s, 2r+s, \dots, 2r+s, u)$  the representation  $T_{Q, \mathbf{e}(r, s, u)}$  has a decomposition  $T_{Q, \mathbf{e}(r, s, u)} = \bigoplus_{j=1}^u T_j \oplus R$  with  $T_j$  indecomposable for all  $j$ ,  $R(n) = 0$ ,

$$T_j(n) = \begin{cases} ke_j & \text{for } j \leq r+s, \\ k(e_j - e_{j'}) & \text{for } r+s < j \leq u \end{cases}$$

Moreover, we have

- for  $u \leq r$ ,  $T_j \simeq X, j = 1, \dots, u$ ,
- for  $r \leq u \leq r+s$ ,  $T_j \simeq \begin{cases} X & \text{for } j = 1, \dots, r, \\ \tilde{Y}_- & \text{for } j = r+1, \dots, u, \end{cases}$
- for  $u \geq r+s$ ,  $T_j \simeq \begin{cases} X & \text{for } j = 1, \dots, 2r+s-u, \\ \tilde{Y}_+ & \text{for } j = 2r+s-u+1, \dots, r, \\ \tilde{Y}_- & \text{for } j = r+1, \dots, u. \end{cases}$

In particular,  $T_{Q, \mathbf{e}(r, s, u)}$  is generic.

*Proof.* In both cases, genericity follows from Corollary ???. The decompositions claimed are just a matter of Linear Algebra and some bookkeeping. As  $T(\alpha_i) = \mathbf{1}_{2r+s}$  for  $i \geq 3$ , except for  $i = n-1$  in the second case, it is sufficient to prove the decomposition claimed only for  $n = 3$  and  $n = 4$ , respectively.

As  $Q$  is of type  $\mathbb{A}_3$  for  $n = 3$ , we know in fact from proposition ?? that  $(\tilde{Y}_+)^r \oplus (\tilde{Y}_-)^{s+r}$  is generic. The numbering of the vertices is not the same we

used in proposition ??, however. In particular, indecomposable representations supported at 3 are not linearly ordered by the existence of non-zero homomorphisms.  $\square$

The following lemma completes the picture for homomorphisms and extensions involving  $X$ :

**Lemma 3.5.** *Let  $U, V, W$  be representations of  $Q'$  with  ${}^1[W, S] = 0, [U, U] = [V, V] = [V, U] = {}^1[U, S] = {}^1[V, S] = 1$  unless  $U = 0$  or  $V = 0$ . for which  $U \oplus Y_+ \oplus Y_- \oplus V \oplus W$  is generic. Suppose that*

- *$U$  and  $Y_+$  are not isomorphic and  $[Y_+, U] = 1$  unless  $U = 0$ ,*
- *$V$  and  $Y_+$  are not isomorphic and  $[V, Y_+] = 1$  unless  $V = 0$ .*

*Then we have that*

- $[X, \tilde{U}] = [\tilde{V}, X] = [S, X] = 1,$
- *the following representations are generic:*
  1.  $\tilde{U} \oplus X \oplus Y_+ \oplus Y_- \oplus V \oplus W,$
  2.  $\tilde{U} \oplus X \oplus \tilde{Y}_- \oplus Y_- \oplus V \oplus W,$
  3.  $\tilde{U} \oplus X \oplus \tilde{Y}_- \oplus \tilde{Y}_+ \oplus V \oplus W.$

*Proof.* We only need to show that

$$[X, \tilde{U}] = [\tilde{V}, X] = 1, {}^1[X, \tilde{U} \oplus W] = {}^1[\tilde{U} \oplus W, X] = 0$$

as all other claims follow from either Lemma ?? or corollary ??. Note that  $[V, Y_-] = [V, Y_+]$  and  $[Y_-, U] = [Y_+, U]$ . All facts claimed are obtained from considering suitable long exact sequences associated to  $\sigma_U, \tau_{Y_+, Y_-}$ , or  $\tau_{Y_-, Y_+}$ .  $\square$

The last claim in the following more technical proposition implies Proposition ?? for any dimension vector  $\mathbf{d}$  with  $d_1 \leq d_2 \leq d_3$ . In addition, the proposition describes the decomposition of a generic representation as a direct sum of indecomposables.

**Proposition 3.6.** 1. For  $n \geq 3$ , there is a decomposition  $d_n = p_n + q_n + 2r_n + s_n + t_n$  with  $p_n, q_n, r_n, s_n, t_n \geq 0$  and a decomposition  $T = T_{Q, \mathbf{d}} = (\oplus_{j=1}^{d_n} T_j) \oplus R$  with  $R(n) = 0$ ,  $T_j$  indecomposable and  $T_j(n) = k$  for all  $j$ ,

$$T_j = \begin{cases} X & \text{for } 1 \leq j - p_n \leq q_n, \\ \tilde{Y}_+ & \text{for } 1 \leq j - p_n - q_n \leq r_n, \\ \tilde{Y}_- & \text{for } 1 \leq j - p_n - q_n - r_n \leq r_n + s_n, \end{cases}$$

and  $T_j$  is not isomorphic to  $\tilde{Y}_+, \tilde{Y}_-$  or  $X$  for other values of  $j$ .

2. For  $n \geq 4$  we have  $r_n \leq r_{n-1}$ . If  $r_n > 0$ ,  $(p_n, q_n, r_n, s_n, t_n)$  satisfies  $s_n = s_{n-1}$  and  $p_n + q_n + r_n = p_{n-1} + q_{n-1} + r_{n-1}$ .
3.  $[T_j, T_l] \leq 1$  and, for  $T_j$  not isomorphic to  $T_l$ , equality holds if and only if  $j > l$  and at least one of the following two conditions is satisfied: either  $T_j \neq \tilde{Y}_-$  or  $T_l \neq \tilde{Y}_+$ , for  $j, l = 1, \dots, d_n$ .
4.  ${}^1[T, T] = 0$ .

*Proof.* For  $n = 3$ , we choose

$$(p_3, q_3, r_3, s_3, t_3) = \begin{cases} (d_3 - d_1 - d_2, 0, d_1, d_2 - d_1, 0) & \text{if } d_1 + d_2 \leq d_3, \\ (0, 0, d_3 - d_2, d_2 - d_1, d_1 + d_2 - d_3) & \text{if } d_1 + d_2 \geq d_3. \end{cases}$$

We set  $R = 0$ , and we let  $T_j$  be the indecomposable direct summand of  $T = T_{Q, \mathbf{d}}$  with

$$T_j(3) = \begin{cases} ke_j & \text{for } 1 \leq j \leq p_3 + q_3 + r_3 + s_3, \\ k(e_j - e_{j'}) & \text{for } 1 \leq j - p_3 - q_3 - r_3 - s_3 \leq r_3, \\ ke_j & \text{for } 1 \leq j - p_3 - q_3 - 2r_3 - s_3, \end{cases}$$

where  $j' = 2d_3 - d_1 - d_2 + 1 - j$ . Note that  $T(\alpha_1)(e_j - e_{j'}) = 0$ . We have

$$\dim T_j = \begin{cases} (0, 0, 1) & \text{for } j = 1, \dots, p_3, \\ (1, 0, 1) & \text{for } j = p_3 + 1, \dots, p_3 + r_3, \\ (0, 1, 1) & \text{for } j = p_3 + r_3 + 1, \dots, p_3 + 2r_3 + s_3, \\ (1, 1, 1) & \text{for } j = p_3 + 2r_3 + s_3 + 1, \dots, p_3 + 2r_3 + s_3 + t_3. \end{cases},$$

and either  $p_3 = 0$  or  $t_3 = 0$ . The representations  $T_j$  satisfy all our conditions by lemma ???. As  $p_3 \cdot t_3 = 0$ , the two indecomposable representations with dimension type  $(0, 0, 1)$  and  $(1, 1, 1)$ , for which there is a non-split extension, cannot both arise. For genericity we could as well have used proposition ???.

We note for further reference that  $p_3 + q_3 + r_3 = d_3 - d_2$ ,  $s_3 = d_2 - d_1$  and that the map  $j \mapsto j'$  is the bijection reversing the numbering from

$$\{j : 1 \leq j - (p_3 + q_3 + r_3 + s_3) \leq r_3\} \rightarrow \{l : 1 \leq l - (p_3 + q_3) \leq r_3\}$$

as we have  $2(p_3 + q_3 + r_3) + s_3 = 2d_3 - d_1 - d_2$ .

For the inductive step, we start from a decomposition of the generic representation  $T_{Q', \mathbf{a}'} = (\oplus_{j=1}^{d_{n-1}} T'_j) \oplus R'$  satisfying the properties listed in our proposition for  $Q'$ . So we know that  $T'_j$  is indecomposable,  $R'(n-1) = 0$ ,  $T'_j(n-1) = k$ ,  $[T'_j, T'_l] \leq 1$  for  $j, l = 1, \dots, d_{n-1}$  and that

$$T'_j = \begin{cases} Y_+ & \text{for } 1 \leq j - p_{n-1} - q_{n-1} \leq r_{n-1}, \\ Y_- & \text{for } 1 \leq j - p_{n-1} - q_{n-1} - r_{n-1} \leq r_{n-1} + s_{n-1}. \end{cases}$$

Moreover, if  $T'_j$  is not isomorphic to  $T'_l$  and if either  $j$  or  $l$  do not lie between  $p_{n-1} + q_{n-1} + 1$  and  $p_{n-1} + q_{n-1} + 2r_{n-1} + s_{n-1}$ , i.e. either  $T'_j$  or  $T'_l$  is not isomorphic to  $Y_+$  or  $Y_-$ , we know that  $[T'_j, T'_l] = 1$  if and only if  $j > l$ .

For  $T_j$  we have either  $\tilde{T}'_j$  or  $X$ , and in order to show that all conditions are satisfied for any pair  $T_j, T_l$ , we use either lemma ??? or a part of lemma ???. We distinguish several cases depending on  $d_n$ . We have

$$R = \begin{cases} (\oplus_{j=d_n+1}^{d_{n-1}} T'_j) \oplus R' & \text{if } d_n \leq d_{n-1}, \\ R' & \text{if } d_n \geq d_{n-1}. \end{cases}$$

In case  $d_n \leq p_{n-1} + q_{n-1}$ , we set  $(p_n, q_n, r_n, s_n, t_n) = (d_n, 0, 0, 0, 0)$ ,  $T_j = \tilde{T}'_j$  for  $j = 1, \dots, d_n$  and apply Lemma ???.

In case  $p_{n-1} + q_{n-1} + 1 \leq d_n \leq p_{n-1} + q_{n-1} + r_{n-1}$  we choose  $(p_n, q_n, r_n, s_n, t_n) = (p_{n-1} + q_{n-1}, d_n - p_{n-1} - q_{n-1}, 0, 0, 0)$  and

$$T_j = \begin{cases} \tilde{T}'_j & \text{for } j = 1, \dots, p_n, \\ X & \text{for } j = p_n + 1, \dots, d_n. \end{cases}$$

In order to verify our conditions for two summands  $T_j, T_l$ , we use lemma ??  $j, l \leq p_n$  and lemma ??, 1. otherwise.

In case  $p_{n-1} + q_{n-1} + r_{n-1} + 1 \leq d_n \leq p_{n-1} + q_{n-1} + r_{n-1} + s_{n-1}$ , we choose  $(p_n, q_n, r_n, s_n, t_n) = (p_{n-1} + q_{n-1}, r_{n-1}, 0, d_n - p_{n-1} - q_{n-1} - r_{n-1}, 0)$  and

$$T_j = \begin{cases} \tilde{T}'_j & \text{for } j = 1, \dots, p_n, \\ X & \text{for } j = p_n + 1, \dots, p_n + q_n, \\ \tilde{T}'_j \simeq \tilde{Y}_- & \text{for } j = p_n + q_n + 1, \dots, d_n. \end{cases}$$

Now we use lemma ??, 2. if one summand under consideration is isomorphic to  $X$  and lemma ?? if not.

Suppose  $p_{n-1} + q_{n-1} + r_{n-1} + s_{n-1} + 1 \leq d_n \leq p_{n-1} + q_{n-1} + 2r_{n-1} + s_{n-1}$ , and set

$$(p_n, q_n, r_n, s_n, t_n) = (p_{n-1} + q_{n-1}, p_{n-1} + q_{n-1} + 2r_{n-1} + s_{n-1} - d_n, d_n - p_{n-1} - q_{n-1} - r_{n-1} - s_{n-1}, s_{n-1}, 0),$$

$T_j = \tilde{T}'_j$  for  $j = 1, \dots, p_n$ , and set  $T^{(1)} = \oplus_{j=1}^{p_n} T_j$ . Then  $T^{(1)}$  does not extend itself, and  $T_1, \dots, T_{p_n}$  satisfy all conditions we claim.

The key point is that  $r_m \geq r_n > 0$  for  $m = 3, \dots, n$  and thus

$$s_n = s_m = s_3 = d_2 - d_1,$$

$$p_n + q_n + r_n = p_m + q_m + r_m = p_3 + q_3 + r_3 = d_3 - d_2$$

for  $m = 3, \dots, n$ . Therefore the representation  $T_{Q, \mathbf{d}}$  has a direct summand  $T^{(2)}$  for which  $T^{(2)}(m)$  is the span of the vectors in the standard basis of  $T^{(2)}(m)$  with indices

$$j = \begin{cases} d_1 - r_{n-1} + 1, \dots, d_1 & \text{for } m = 1, \\ d_3 - d_2 + 1, \dots, d_3 - d_1 + r_{n-1} & \text{for } m = 2, \\ d_3 - d_2 - r_{n-1} + 1, \dots, d_3 - d_1 + r_{n-1} & \text{for } m = 3, \dots, n-1, \\ d_3 - d_2 - r_{n-1} + 1, \dots, d_3 - d_1 + r_n & \text{for } m = n. \end{cases}$$

Indeed, the map  $j \mapsto j'$  is the bijection reversing the numbering from

$$\{d_3 - d_1 + 1, \dots, d_3 - d_1 + r_{n-1}\} \rightarrow \{d_3 - d_2 - r_{n-1} + 1, \dots, d_3 - d_2\}.$$

We remark that  $T^{(2)}$  is just the representation  $T_{Q, \mathbf{e}(r_{n-1}, s_{n-1}, u)}$  for  $u = d_n - p_n = d_2 - d_1 + r_{n-1} + r_n$ , which by lemma ?? is isomorphic to  $\oplus_{j=p_n+1}^{d_n} T_j$  with

$$T_j = \begin{cases} X & \text{for } 1 \leq j - p_n \leq q_n, \\ \tilde{Y}_+ & \text{for } 1 \leq j - p_n - q_n \leq r_n, \\ \tilde{Y}_- & \text{for } 1 \leq j - p_n - q_n - r_n \leq r_n + s_n. \end{cases}$$

By corollary ??,  $T^{(2)}$  does not extend itself, and its indecomposable direct summands  $T_{p_n+1}, \dots, T_{p_n+q_n+2r_n+s_n}$  satisfy the conditions claimed. By lemma ?? and lemma ??, 3., the summands  $T_1, \dots, T_{p_n+q_n+2r_n+s_n}$  of  $T = T^{(1)} \oplus T^{(2)}$  satisfy all conditions as well.

In case  $1 \leq d_n - p_{n-1} - q_{n-1} - 2r_{n-1} - s_{n-1} \leq t_{n-1}$  we set

$$(p_n, q_n, r_n, s_n, t_n) = (p_{n-1} + q_{n-1}, 0, r_{n-1}, s_{n-1}, d_n - p_{n-1} - q_{n-1} - 2r_{n-1} - s_{n-1}),$$

$T_j = \tilde{T}'_j$  for  $j = 1, \dots, d_n$ . Finally, in case  $d_n \geq d_{n-1}$ , we take

$$(p_n, q_n, r_n, s_n, t_n) = (p_{n-1} + q_{n-1}, 0, r_{n-1}, s_{n-1}, t_{n-1} + d_n - d_{n-1})$$

and

$$T_j = \begin{cases} \tilde{T}'_j & \text{if } j \leq d_{n-1}, \\ S & \text{if } j > d_{n-1}, \end{cases}$$

and we conclude in both cases by lemma ??.  $\square$

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